L21 MUTUAL INDUCTANCE + LC + LCR CIRCUITS

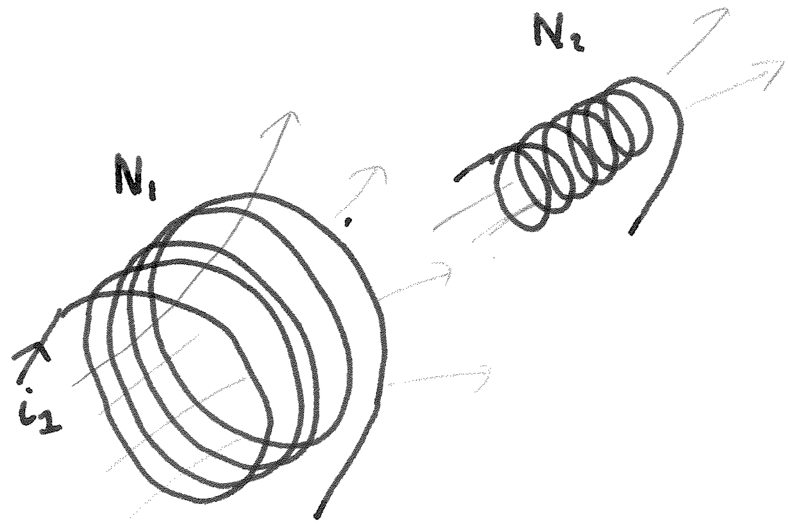
Today, after returning briefly to the topic of inductance, we will look at oscillatory "LC" circuits. You will see that these are the electrical analog of a weight on a spring, or a pendulum.

30.1 Mutual Inductance

Just as a coil has a self inductance, it can also induce an e.m.f in any other electrical coil which couples to its magnetic field. We define

$$M_{21} = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_1 \Phi_{B1}}{i_2}$$

as the mutual inductance between coil two and the current in coil 1.



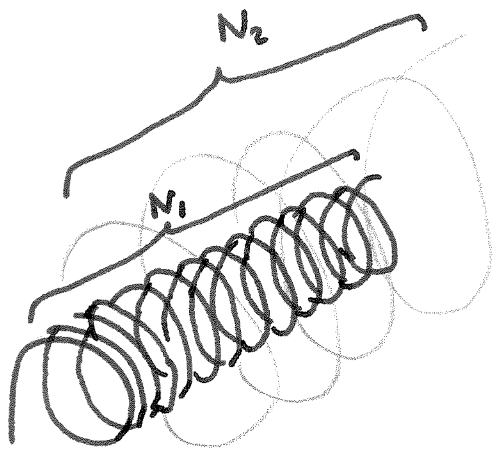
$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{B2}}{dt} = -M_{21}\frac{di_1}{dt}$$

It turns out that the mutual inductance of 1 from 2 is equal to the mutual inductance of 2 from 1, so that $M_{21} = M_{12}$ $\left(= \partial^2 U_B / \partial i_1 \partial i_2\right)$

$$M = M_{12} = M_{21}$$

$$\mathcal{E}_2 = -M\frac{di_1}{dt} \qquad \mathcal{E}_1 = -M\frac{di_2}{dt}.$$

e.g Mutual inductance of two coils



$$B_1 = \mu_0 n_1 i_1 = \mu_0 \frac{N_1}{L} i_1$$

$$M = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_2 B_1 A}{i_1} = \frac{N_2 \mu_0 N_1 A}{L} = \frac{\mu_0 N_1 N_2 A}{L}$$

e.g $l = 0.5\ m$, $A = 10\ cm^2 = 10^{-3}\ m^2$, $N_1 = 1000$, $N_2 = 10$

$$M = \frac{(4\pi\times10^{-7}\ Wb/A.m)(10^{-3})(1000)(10)}{0.5} = 8\pi\times10^{-6}\ H$$

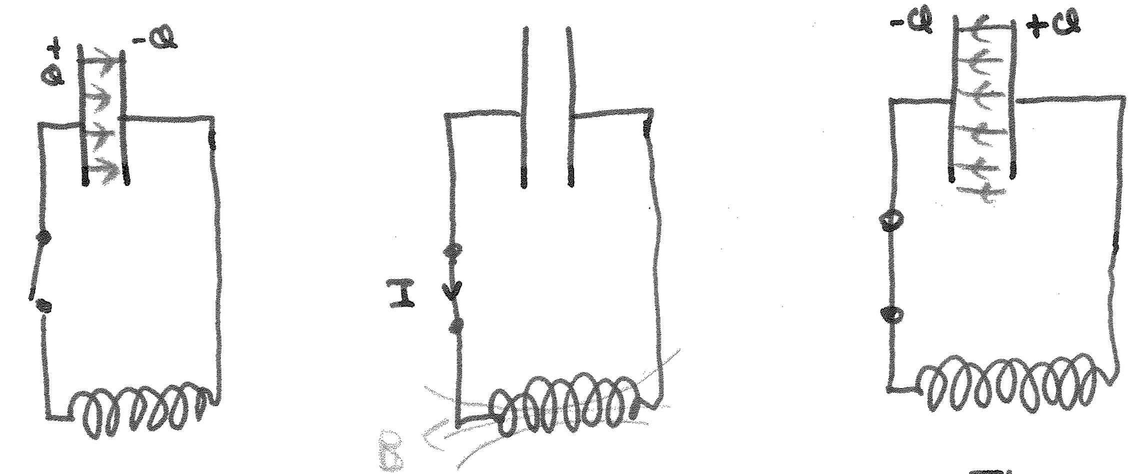
$$\equiv \underline{25\ \mu H}$$

30.5 LC circuit

Suppose we take a charged capacitor with charge $Q = CV$ and connect its plates together with a coil. Current will flow through the coil & charge will move from one plate to the other. If there is no resistance then the total energy

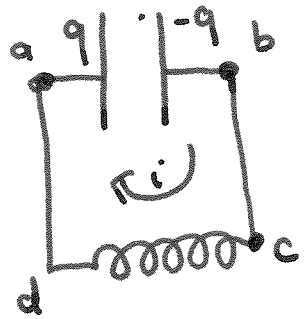
$$U = \underbrace{\frac{1}{2} L i^2}_{u_B} + \underbrace{\frac{1}{2} \frac{Q^2}{C}}_{u_E}$$

will be conserved.



0 T/4 T/2

Kirchoff's law: LC circuit



$$V_{ab} = \frac{q}{C} \qquad V_{cd} = L\frac{di}{dt}$$

$$-\frac{q}{C} - L\frac{di}{dt} = 0 \quad \Rightarrow \quad \frac{di}{dt} + \frac{q}{LC} = 0$$

$$i = \frac{dq}{dt} \Rightarrow \frac{di}{dt} = \frac{d^2q}{dt^2}$$

$$\boxed{\frac{d^2q}{dt^2} + \frac{q}{LC} = 0}$$

Recall $$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$

$$x = A\cos[\omega t + \phi]$$

$$\frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \omega^2 = k/m, \quad \omega = \sqrt{\frac{k}{m}}$$

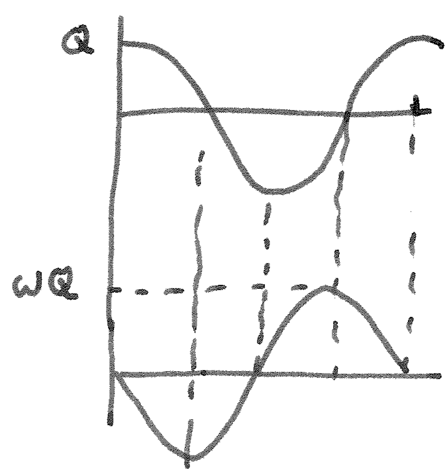
Now
$$\boxed{\begin{aligned} q &= Q\cos(\omega t + \phi) \\ i = \frac{dq}{dt} &= -\omega Q \sin(\omega t + \phi) \end{aligned}}$$

$$\frac{d^2q}{dt^2} = -\omega^2 q \Rightarrow \omega^2 = \frac{1}{LC}$$

$$\boxed{\omega = \frac{1}{\sqrt{LC}}}$$

$f = \omega/2\pi$

OSCILLATION FREQ



Energy conservation. Spring $\frac{1}{2}mv^2 + \frac{kx^2}{2} = U = \frac{1}{2}kA^2$

$$v = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

LC circuit $i = \pm\frac{1}{\sqrt{LC}}\sqrt{Q^2 - q^2}$.

Mass-Spring	LC circuit
$KE = \frac{1}{2}mv^2$	$M.E = \frac{1}{2}Li^2$
$PE = \frac{1}{2}kx^2$	$E.E = q^2/2C$
$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$	$\frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{Q^2}{C}$.
$v = \pm\sqrt{k/m}\sqrt{A^2 - x^2}$	$i = \pm\frac{1}{\sqrt{LC}}\sqrt{Q^2 - q^2}$
$V_x = dx/dt$	$i = dq/dt$
$\omega = \sqrt{\frac{k}{m}}$	$\omega = \frac{1}{\sqrt{LC}}$
$x = A\cos(\omega t + \phi)$	$q = Q\cos(\omega t + \phi)$.

$$\text{Energy} = \frac{1}{2} L i^2 + \frac{q^2}{2C} = U = \frac{Q^2}{2C}$$

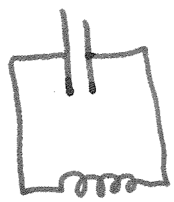
$$i^2 = \frac{1}{L}\left(\frac{Q^2 - q^2}{C}\right) = \frac{1}{LC}\left(Q^2 - q^2\right).$$

$$i = \frac{1}{\sqrt{LC}} \sqrt{Q^2 - q^2}.$$

e.g. 25 μF charged up by 300V power supply, connected to a 10 mH inductor.

a) Find f & T

b) q(t) i(t) @ t= 1.2ms



a) $\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-2} \times 25 \times 10^{-6}}} = \frac{1}{5 \times 10^{-4}}$ rad/s.

$f = \frac{\omega}{2\pi} = \frac{2000}{2\pi} = \underline{320Hz}$

$T = \frac{1}{f} = \frac{1}{320Hz} = 3.1 \times 10^{-3} = \underline{3.1ms}$

b) $Q = CV = 25 \times 10^{-6} \times 3 \times 10^{2} = 7.5 \times 10^{-3} C = \underline{7.5 mC}$

$q(t) = 7.5 \times 10^{-3} \cos(2 \times 10^{3} t)$

$i = -7.5 \times 10^{-3} \times 2 \times 10^{3} \sin(2000t) = -15 \sin(2000t)$

$t = 1.2\,ms$

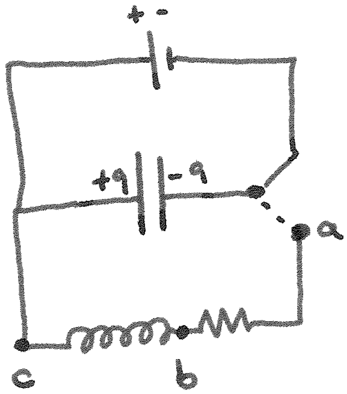
$$q(t) = 7.5 \times 10^{-3} \cos(2.4)$$

$$= \underline{-5.5 \times 10^{-3}\,C}$$

$$i(t) = -15 \sin(2.4) = \underline{\underline{-10\,A}}\,.$$

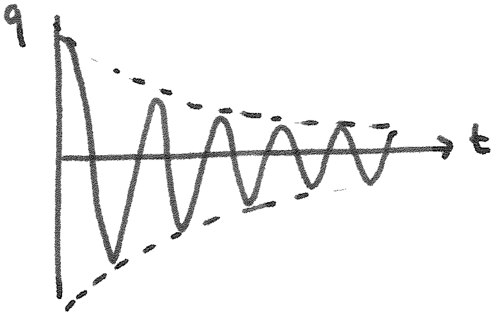
21.12

30.6 Damped oscillations: LCR circuit



Resistance ~ friction

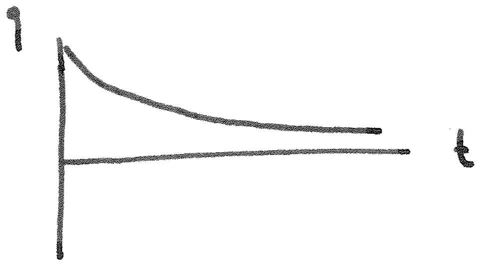
(i^2R losses in resistor)



a) underdamped. $R < 2\sqrt{\frac{L}{C}}$.

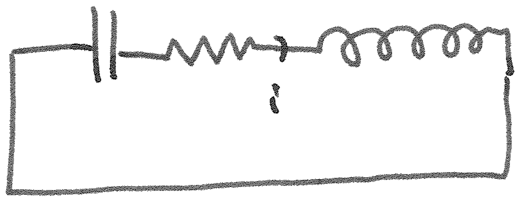
b) Critically damped.

$R = 2\sqrt{\frac{L}{C}}$



c) Overdamped

$R > 2\sqrt{\frac{L}{C}}$.



$$-\frac{q}{C} - iR - L\frac{di}{dt} = 0$$

$$\boxed{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0}$$

c.f. damped harmonic oscillator.

$R=0 \Rightarrow \equiv LC$ circuit.

general soln. $\left(R < 2\sqrt{L/C}\right)$

$$\boxed{q = Q e^{-R/2L\, t} \cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\; t + \phi\right)}$$

$$\tilde{\omega} = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \rightarrow 0 \text{ as } R\downarrow$$

p.s note $\tilde{\omega} = \sqrt{\omega_0^2 - \frac{1}{(2\tau)^2}}$

e.g. a) What R is required so that

$\omega = \omega_0/2$?

$$\omega = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}} = \frac{\omega_0}{2}$$

$$\omega_0^2 - R^2/4L^2 = \omega_0^2/4$$

$$(R^2/4L^2) = 3\omega_0^2/4$$

$$R^2 = 3\omega_0^2 L^2 = \frac{3L}{C}$$

$$R = \sqrt{\frac{3L}{C}}$$

b) If $f_0 = 4\,\text{MHz}$ & $L/R = 1\mu s$ calculate f.

$$f = 2\pi\sqrt{\omega_0^2 - (1/2\tau)^2} = \sqrt{f_0^2 - \left(\frac{2\pi}{2\tau}\right)^2}$$

$$= \sqrt{(4\times10^6)^2 - \left(\frac{3.14}{10^{-6}}\right)^2}$$

$$= 10^6 \times \sqrt{6.13} = 2.47\times10^6\,\text{Hz}$$

$$= \underline{2.47\,\text{MHz}}$$