L20. INDUCTANCE

Just as a moving body has an inertia – a resistance to a changing momentum, electric currents also have a resistance to change.

When we send a current through a coil we build up a field in space, and the field contains energy. The larger the field energy, the more the circuit resists any attempt to change the current.

This "stubborn resistance to change" is associated with the phenomenon of inductance.

We'll see how "inductance" is responsible for forcing a current through the spark plug in a car's ignition system.

SELF INDUCTANCE

The self inductance of a coil is defined as the ratio of the magnetic flux to the current

$$L = \left(\frac{N \Phi_B}{i} \right) \qquad \text{unit: Henry (H)}$$

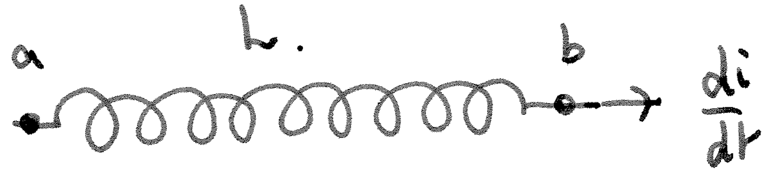
From this we see that

$$N \frac{d\Phi_B}{dt} = \text{rate of change of flux}$$

$$= L \frac{di}{dt} \Rightarrow \boxed{\mathcal{E} = -L \frac{di}{dt}}$$

If we put an inductor in a circuit it causes an e.m.f around the circuit. Although the field circulating around the circuit is not conservative, it turns out that there is a conservative component, and the voltage drop across an inductor is given by

$$\boxed{V_{ab} = V_a - V_b = L\frac{di}{dt}}$$

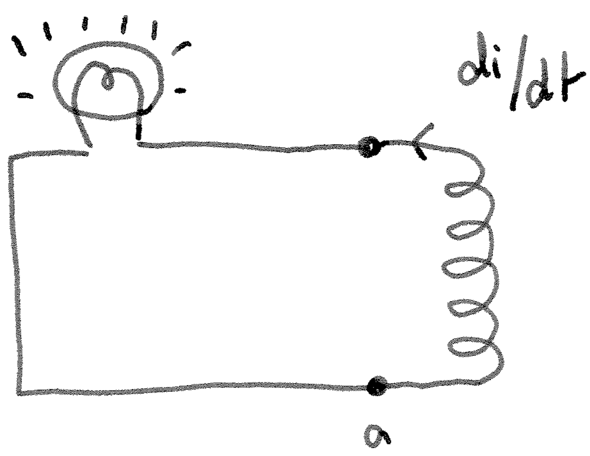


To understand this we need to divide the field into a conserved component E_c & a non-conserved component E_n. The loop integral of E_c is zero, so

$$\oint \vec{E}.dl = \int (E_c + E_n).dl = \int E_n.dl$$

Now the non-conservative part of the field is concentrated inside the solenoid, so

$$\int_{a\ \text{SOLENOID}}^{b} E_n.dl = -L\,{di}/{dt}$$

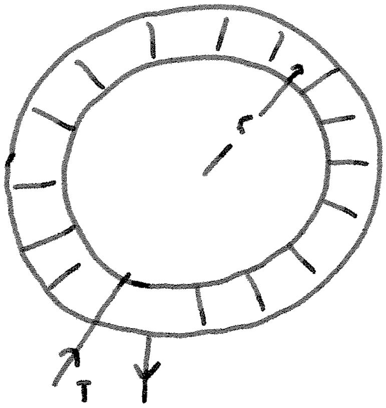


But since the solenoid has a low resistance $E \simeq 0$ in the solenoid, so that

$\vec{E}_c = -\vec{E}_n$ inside the solenoid, thus

$$\int_{a\ \text{solenoid}}^{b} \vec{E}_c \cdot d\vec{\ell} = L\,di/dt = V_a - V_b.$$

e.g. Self-inductance of a toroidal solenoid



$$\Phi_B = BA = \frac{\mu_0 N i A}{2\pi r}$$

$$L = \frac{N\Phi_B}{i} = \frac{\mu_0 N^2 A}{2\pi r}$$

e.g. $N = 200$ turns, $A = 5.0\,cm^2 = 5\times10^{-4}\,m^2$, $r = 0.1\,m$

$$L = \frac{(4\pi\times10^{-7}\,Wb/A\cdot m)(200)^2\times(5\times10^{-4})}{2\pi\,(0.1)} = 40\times10^{-6}\,H$$

$$= 40\,\mu H.$$

e.g. suppose I increases from $I = 0$ to $I = 6A$ in $3\mu s$, calculate the induced e.m.f.

$$|\varepsilon| = L\frac{di}{dt} = 40\times10^{-6}\times\frac{6}{3\times10^{-6}} = \underline{\underline{80\,V}}$$

30.3. Magnetic Field Energy

We saw previously that the energy per unit volume of the electric field was

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

Let us calculate the analogous quantity for the magnetic field in an inductor. Energy supplied by external source in building up the field in an inductor is

$$U = \int_0^T \overbrace{P\,dt}^{du} = \int_0^T \overbrace{V_{ab}\, i\, dt}^{du} = L\int_0^T \overbrace{i\frac{di}{dt}\,dt}^{du}$$

We see that $du = Li\,di$, so

$$U = L\int_0^I i\,di = \frac{LI^2}{2}$$

This energy does not depend on how the current is increased – whether I is turned on quickly, or slowly.

Now lets look at the energy per unit volume

$$u_B = \frac{U}{2\pi r A} = \frac{LI^2}{4\pi r A}$$

but $L = \dfrac{\mu_0 N^2 A}{2\pi r}$

so

$$u_B = \frac{1}{2} \frac{\mu_0 N^2 I^2}{(2\pi r)^2}$$

But $B = \dfrac{\mu_0 N I}{2\pi r}$ inside the solenoid, so

$$\boxed{u_B = \frac{1}{2\mu_0} B^2}$$

e.g. A high field magnet has a working volume of height 1m, radius 2cm. Calculate the field energy if B = 50 T.

$$U_B = \frac{1}{2\mu_0} B^2 \times \pi r^2 h = \frac{(50\,T)^2 \times \cancel{\pi} \times (0.02)^2 \times 1}{8\cancel{\pi} \times 10^{-7}}$$

$$= 1.25 \times 10^6\,J = 1.25\,MJ \simeq 2 \text{ sticks of dynamite.}$$

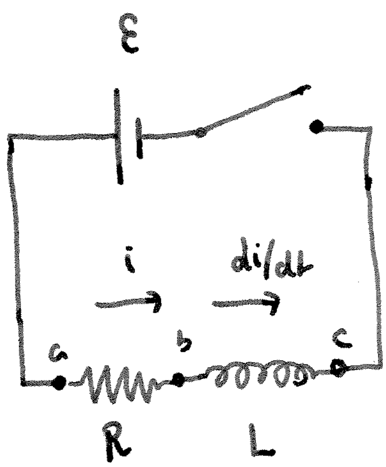
1 stick of dynamite $\approx$ 0.5 MJ

e.g. What inductance is needed to store 1 kWh in a coil containing a current of 200A.

$$\frac{1}{2} L I^2 = 10^3 \times 3.6 \times 10^3 \text{ J}$$

$$L = \frac{2 \times 10^6 \times 3.6}{(200)^2} = \underline{180 \text{H}}.$$

30.4 LR circuit



$$\left.\begin{aligned} V_{ab} &= iR \\ V_{bc} &= L\, di/dt \end{aligned}\right\}$$

At $t=0$ $i=0$ – requires time to build up

$$\mathcal{E} - iR - L\,di/dt = 0$$

$$\frac{di}{dt} = \frac{\mathcal{E} - iR}{L} = \frac{\mathcal{E}}{L} - \frac{R}{L} i$$

At $t=0$, $i=0$

$$\frac{di}{dt} = \frac{\mathcal{E}}{L}$$

greater L, the more slowly i increases.

At $t \to \infty$, $i = \mathcal{E}/R$, $di/dt = 0$.

$$\frac{di}{dt} = -\frac{R}{L}\left(i - \frac{\mathcal{E}}{R}\right)$$

$$\frac{di}{(i - \mathcal{E}/R)} = -\frac{R}{L}\,dt$$

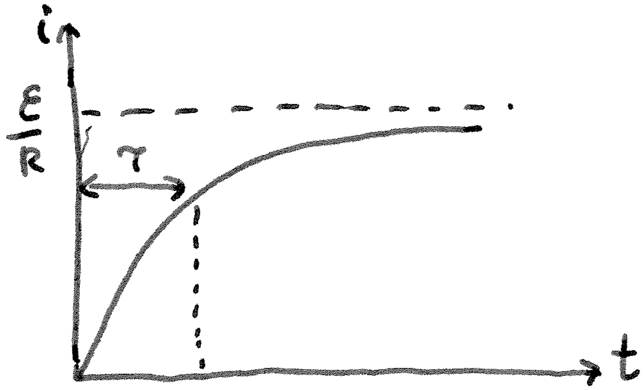
$$\int_0^i \frac{di'}{i' - (\mathcal{E}/R)} = -\frac{R}{L}\int_0^t dt'$$

$$\ln\left|\frac{i - (\mathcal{E}/R)}{-\mathcal{E}/R}\right| = -\frac{R}{L}\,t$$

$$\frac{i - \mathcal{E}/R}{-\mathcal{E}/R} = e^{-t(R/L)}$$

$$\boxed{\tau = L/R}$$

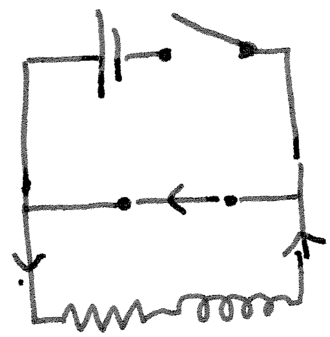
$$\boxed{i = \mathcal{E}/R\left(1 - e^{-t/\tau}\right)}$$



e.g. $R = 1\Omega$, $L = 1mH$ $\quad \tau = \frac{1mH}{1\Omega} = \underline{\underline{1ms}}$

Discharge of inductor

$i = i_0 = \frac{\mathcal{E}}{R}$ initially



$$iR + L\frac{di}{dt} = 0$$

$$\frac{di}{i} = -\frac{L}{R}\,dt$$

$$\int_{i_0}^{i} \frac{di'}{i'} = -\frac{L}{R}t = \ln\left(\frac{i}{i_0}\right)$$

$$\rightarrow \boxed{i = i_0 e^{-t/\tau}}$$

e.g. Device of $R = 175\,\Omega$, operates + steady state current $i_0 = 36\,mA$. Is to connected in series with an inductor so that the current can not rise more than $4.9\,mA$ in the first $58\,\mu s$.

$$i = i_0\left(1 - e^{-t/\tau}\right) \qquad \tau = \frac{L}{R}$$

$$\left(1 - \frac{i}{i_0}\right) = e^{-t/\tau} \qquad t = 58\,\mu s$$

$$-\frac{t}{\tau} = \ln\left(1 - i/i_0\right)$$

$$\tau = \frac{-t}{\ln\left(1 - i/i_0\right)} = \frac{L}{R}$$

$$L = -\frac{Rt}{\ln\left(1 - i/i_0\right)} = \frac{-175\,\Omega \times 58 \times 10^{-6}}{\ln\left(1 - \frac{4.9}{36}\right)} = \underline{69\,mH}$$

$$\tau = \frac{69\,mH}{175\,\Omega} = 3.9 \times 10^{-4}\,s = 390\,\mu s.$$

e.g. Calculate how the energy in an inductor decays

with time in an L-R circuit

$$U_B = \frac{1}{2} L I^2$$

$$I = I_0 e^{-t/\tau} \qquad \tau = L/R$$

$$\left.\begin{array}{l} \\ \end{array}\right\} \quad U_B = \frac{1}{2} L I_0^2 e^{-2t/\tau}$$

$$= U_{B0} e^{-2t/\tau}.$$

$$= U_{B0} e^{-2tR/L}.$$