L20. **INDUCTANCE**

Just as a moving body has an inertia—a resistance to changing momentum, electric currents also have a resistance to change.

When we send a current through a coil, we build up a field in space, and the field contains energy. The larger the field energy, the more the circuit resists any attempt to change the current.

This "stubborn resistance to change" is associated with
the phenomenon of inductance.

We'll see how "inductance" is responsible for forcing a current through the spark plug in a car's ignition system.
SELF INDUCTANCE

The self inductance of a coil is defined as the ratio of the magnetic flux to the current.

\[ L = \left( \frac{N \Phi_B}{i} \right) \quad \text{unit: Henry (H)} \]

From this we see that

\[ \frac{N \Phi_B}{dt} = \text{rate of change of flux} \]

\[ = L \frac{di}{dt} \Rightarrow E = -L \frac{di}{dt} \]
If we put an inductor in a circuit it causes an e.m.f around the circuit. Although the field circulating around the circuit is not conservative, it turns out that there is a conservative component, and the voltage drop across an inductor is given by

\[ V_{ab} = V_a - V_b = L \frac{di}{dt} \]
To understand this we need to divide the field into a conserved component $E_c$ & a non-conserved component $E_n$. The loop integral of $E_c$ is zero, so

$$\oint E\cdot dl = \oint (E_c + E_n)\cdot dl = \oint E_n\cdot dl$$

Now the non-conservative part of the field is concentrated inside the solenoid, so

$$\oint E_n\cdot dl = -L \frac{di}{dt}$$

But since the solenoid has a low resistance $E=0$ in the solenoid, so that
\[ E_c = -E_n \text{ inside the solenoid, thus} \]

\[ \oint_{\text{solenoid}} E_c \cdot d\ell = \frac{d}{dt} (L \frac{di}{dt}) = V_a - V_b. \]
Self-inductance of a toroidal solenoid

\[ \Phi_B = \oint_A \mathbf{B} \cdot d\mathbf{A} = \frac{\mu_0 N}{2\pi r} I \]

\[ L = \frac{N}{i} \Phi_B = \frac{\mu_0 N^2 A}{2\pi r} \]

\[ L = \frac{(4\pi \times 10^{-7} \text{ Wb/A-m}) (200)^2 \times (5 \times 10^{-4})}{2\pi (0.1)} = 40 \times 10^{-6} \text{ H} \]

\[ = 40 \mu\text{H} \]

\[ \text{e.g. Suppose I increases from } I = 0 \text{ to } I = 6 \text{ A in 3 ms, calculate the induced e.m.f.} \]

\[ \left| E \right| = L \frac{di}{dt} = \frac{40 \times 10^{-6} \times 6}{3 \times 10^{-6}} = 80 \text{ V} \]
30.3. **Magnetic Field Energy**

We saw previously that the energy per unit volume of the electric field was

\[ U_E = \frac{1}{2} E_0 E^2 \]

Let us calculate the analogous quantity for the magnetic field in an inductor. Energy supplied by external source in building up the field in an inductor is

\[ U = \int_0^T \frac{du}{P dt} = \int_0^T \frac{du}{V_{ab} i dt} = \int_0^T \frac{du}{L \frac{di}{dt}} \]
We see that \( du = L i \, di \), so

\[
U = \int_0^1 \frac{L \, i \, di}{2} = \frac{L I^2}{2}
\]

This energy does not depend on how the current is increased—whether it is turned on quickly, or slowly.

Now lets look at the energy per unit volume

\[
U_B = \frac{U}{2\pi r A} = \frac{L I^2}{4\pi r A}
\]

but \( L = \frac{M_0 N^2 A}{2\pi r} \)
So

\[ U_B = \frac{1}{2} \frac{\mu_0 N^2 I^2}{(2\pi r)^2} \]

But \[ B = \frac{\mu_0 NI}{2\pi r} \] inside the solenoid, so

\[ U_B = \frac{1}{2} \mu_0 B^2 \]

**Example:** A high field magnet has a working volume of height 1 m, radius 2 cm. Calculate the field energy if \( B = 5 \text{T} \).

\[ U_B = \frac{1}{2} \frac{B^2 \times \pi r^2 h}{2\mu_0} = \frac{(5\text{T})^2 \times \pi \times (0.02)^2 \times 1}{8\pi \times 10^{-7}} \]

\[ = 1.25 \times 10^6 \text{ J} = 1.25 \text{ MJ} \approx 2 \text{ slabs of dynamite} \]

1 slab of dynamite = 0.5 MJ
e.g. What inductance is needed to store 1 kWh in a coil containing a current of 200 A.

\[ \frac{1}{2} LI^2 = 10^3 \times 3.6 \times 10^3 \, J \]

\[ L = \frac{2 \times 10^6 \times 3.6}{(200)^2} = 180 \, H. \]
30.4 **LR circuit**

\[ V_{ab} = iR \]
\[ V_{bc} = L \frac{di}{dt} \]

At \( t=0 \), \( i=0 \) - requires time to build up

\[ E - iR - L \frac{di}{dt} = 0 \]

\[ \frac{di}{dt} = \frac{E - iR}{L} = \frac{E}{L} - \frac{R}{L} i \]

At \( t=0 \), \( i=0 \)

\[ \frac{di}{dt} = \frac{E}{L} \]

greater \( L \), the more slowly \( i \) increases.
At $t \to \infty$, $i = \frac{E}{R}$, $\frac{di}{dt} = 0$.

\[
\frac{di}{dt} = -\frac{R}{L} \left( i - \frac{E}{R} \right)
\]

\[
\frac{di}{i - (\frac{E}{R})} = -\frac{R}{L} \int_{t}^{t'} dt'
\]

\[
\ln \left( \frac{i - (\frac{E}{R})}{-\frac{E}{R}} \right) = -\frac{R}{L} t
\]

\[
\frac{i - \frac{E}{R}}{-\frac{E}{R}} = e^{-t \frac{L}{R}}
\]

\[
i = \frac{E}{R} \left( 1 - e^{-t/\tau} \right)
\]

[Graph showing the current $i$ as a function of time $t$, with a time constant $\tau = L/R$]
\[ R = 1 \Omega, \quad L = 1 \text{mH} \quad \Rightarrow \quad T = \frac{1 \text{mH}}{1 \Omega} = 1 \text{ms} \]

**Discharge of inductor**

\[ i = i_0 = \frac{E}{R} \text{ initially} \]

\[ \tau R + L \frac{di}{dt} = 0 \]

\[ \frac{di}{i} = -\frac{L}{R} dt \]

\[ \int_{i_0}^{i} \frac{di'}{i'} = -\frac{L}{R} t = \ln\left(\frac{i}{i_0}\right) \]

\[ \Rightarrow \quad i = i_0 e^{-\frac{t}{\tau}} \]
Device of $R = 175\, \Omega$, operates in steady state current $i_0 = 36\, mA$. It is connected in series with an inductor so that the current can not rise more than $4.9\, mA$ in the first $58\, \mu s$.

\[ i = i_0 (1 - e^{-t/\tau}) \quad T = \frac{L}{R} \]

\[ (1 - \frac{i}{i_0}) = e^{-t/\tau} \quad t = 58\, \mu s \]

\[ -\frac{t}{\tau} = \ln(1 - \frac{i}{i_0}) \]

\[ T = \frac{-t}{\ln(1 - \frac{i}{i_0})} = \frac{L}{R} \]

\[ L = -\frac{Rt}{\ln(1 - \frac{i}{i_0})} = \frac{-175\, \Omega \times 58 \times 10^{-6}}{\ln(1 - \frac{4.9}{36})} = 69\, m\mu H \]

\[ T = \frac{69\, m\mu H}{175\, \Omega} = 3.9 \times 10^{-4} \, s = 390\, \mu s. \]
e.g. Calculate how the energy in an inductor decays into heat in an L-R circuit.

\[ U_B = \frac{1}{2} LI^2 \]

\[ I = I_0 e^{-t/\tau} \]

\[ \tau = LR \]

\[ U_B = \frac{1}{2} L (I_0 e^{-2t/\tau})^2 = U_{B0} e^{-2t/\tau} \]

\[ = U_{B0} e^{-2tR/L} \]