Sources of Magnetic Field.

We've seen how the magnetic field exerts a force on moving charges. Now we need to understand how a moving charge gives rise to a magnetic field. We'll start with the case of a single charge, then we will look at the field produced by a small segment of current, and this will lead us finally to our next Maxwell equation—"Ampere's law."
28.1 Field from a moving charge.

\[ B = \frac{\mu_0}{4\pi} \frac{|\mathbf{v} \times \mathbf{r}|}{r^2} \]

\( (\mu_0 = 4\pi \times 10^{-7} \text{ Tm/A}) \)

In vector notation

\[ \mathbf{B} = \frac{\mu_0}{4\pi} q \left( \frac{\mathbf{v} \times \mathbf{r}}{r^2} \right) \]

Units:
- \([B] = 1 \text{T}\)
- \([\mu_0] = 1 \text{T}/(\text{m}^{-1}\text{s}^{-1}) = 1 \text{T/mA}\)

Why is \(\mu_0 = 4\pi \times 10^{-7}\) exactly? We'll see later that this
derives from the definition of the Ampere. Turns out that

\[ \frac{1}{\mu_0} = c^2 \Rightarrow \frac{1}{4\pi \epsilon_0} = \frac{\mu_0 c^2}{4\pi} = 10^{-7}c^2 = 9 \times 10^9 \text{ Nm}^2/\text{C} \]
Comparison of electric + magnetic force between two protons

\[ \vec{F}_E = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r^2} \hat{j} \]

\[ \vec{B} = \frac{\mu_0 q}{4\pi} \frac{\vec{v} \times \hat{r}}{r^2} \]

\[ = \frac{\mu_0 q v}{4\pi r^2} \hat{r} \times \hat{j} \]

\[ \vec{F}_B = q (-\vec{v}) \times \vec{B} = \frac{q (-\vec{v}) \times \vec{B}}{4\pi \varepsilon_0} \hat{k} = \frac{\mu_0 q v^2}{4\pi r^2} \hat{j} \]

\[ \frac{\vec{F}_B}{\vec{F}_E} = \frac{\mu_0 e_0 v^2}{\varepsilon_0} = \frac{V^2}{c^2} \]

When \( v \ll c \), \( \frac{F_B}{F_E} < 1 \) - magnetic force only comparable with electric force when \( v \approx c \), the speed of light.
Field of a current element

\[ d\mathbf{B} = \left( \frac{\mu_0}{4\pi} \right) \frac{nqvdA \, d\ell}{r^2} = \frac{\mu_0}{4\pi} \frac{I \, d\ell \sin \theta}{r^2} \]

"Biot-Savart law"

\[ \mathbf{B} = \int d\mathbf{B} = \frac{\mu_0}{4\pi} \int I \, d\ell \times \hat{r} \]

Can only measure net field from entire circuit and this can be used to indirectly verify the Biot-Savart law.
Calculate field produced by a 1cm segment of wire, 2m away (a) at P₁, straight to the side & (b) P₂ at 30° to the segment, if \( I = 125 \text{A} \)

![Diagram showing a wire segment with points P₁ and P₂, and a 30° angle from the segment]

a) \[ B_{P₁} = \frac{\mu_0}{4\pi} \frac{125 \cdot (0.01) \cdot 180}{2} = 10^{-7} \cdot \frac{125 \cdot 10^{-2}}{4} \]

\[ = 31.25 \times 10^{-9} \]

\[ = 3.13 \times 10^{-8} \text{ N} \]

b) \[ B_{P₂} = 10^{-7} \cdot \frac{125 \cdot 10^{-2}}{4} \cdot \sin 30° = 1.56 \times 10^{-8} \text{ N} \]
\[ dB = \frac{\mu_0 I \, dl}{4\pi x^2 + R^2} \sin \phi \]

\[ \sin \phi = \frac{R}{\sqrt{x^2 + R^2}} \]

\[ B = \frac{\mu_0 I R}{4\pi} \int_{-a}^{a} dx \frac{1}{(x^2 + R^2)^{3/2}} = \frac{\mu_0 I R}{4\pi} \left[ \frac{2}{R^2 (x^2 + R^2)^{1/2}} \right]_{-a}^{a} \]

\[ = \frac{\mu_0 I}{2\pi} \frac{a}{(R^2 + a^2)^{1/2}} R \]

\[ a \to \infty \]

\[ B = \frac{\mu_0 I}{2\pi R} \]

\[ \text{infinite line} \]
Calculate field midway between two wires carrying $I$ in opposite directions, separation $2d$.

\[ B_1 = \frac{\mu_0 I}{2\pi d} \]
\[ B_2 = -\frac{\mu_0 I}{2\pi d} \quad \text{(opposite currents)} \]

\[ B = \frac{\mu_0 I}{2\pi d} \]
28.4 Force between two wires

\[ B = \frac{\mu_0 I_1}{2\pi r} \]

\[ F = I_2 l B \]

\[ = I_2 l \frac{\mu_0 I_1 I_2}{2\pi r} \]

1 Ampere = \(2 \times 10^{-7} \text{ N} \cdot \text{m/A} \) at 1 m separation.

\[ 2 \times 10^{-7} = \frac{\mu_0 (1A)^2}{2\pi \times 1\text{m}} \Rightarrow \mu_0 = 4\pi \times 10^{-7} \]
28.5 Field of circular current loop.

\[ \delta B = \frac{\mu_0 I}{4\pi} \frac{\delta l}{(x^2 + a^2)^\frac{3}{2}} \]

\[ \delta B_x = \delta B \cos \phi = \frac{\delta B}{(x^2 + a^2)^\frac{3}{2}} a \]

\[ B_x = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi a} \frac{dl}{(x^2 + a^2)^\frac{3}{2}} = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + x^2)^\frac{3}{2}} \]

N loops:

\[ B_x = \frac{\mu_0 NI}{2} \frac{a^2}{(a^2 + x^2)^\frac{3}{2}}. \]